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# Plana's Summation Formula for Holomorphic Functions of Exponential Type (Microlocal Analysis and Related Topics)

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# Plana's Summation Formula for Holomorphic Functions of Exponential Type

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## Abstract

In this paper we derive Plana's summation formula for holomorphic functions of exponential type by using theory of analytic functionals with unbounded carrier.

In §1, we will recall the definitions and properties of analytic functionals with unbounded carrier.

In §2, we derive Plana's summation formula for holomorphic functions of exponential type.

In §3, we will give some examples of Plana's summation formula.

## 1 The definitions of analytic functionals with unbounded carrier and their properties.

Let  $L$  and  $L_\varepsilon$  be following strip regions:

$$\begin{aligned} L &= (-\infty, a] + i[-b, b], \\ \text{For } \varepsilon > 0, \quad L_\varepsilon &= (-\infty, a + \varepsilon] + i[-b - \varepsilon, b + \varepsilon]. \end{aligned}$$

For  $\varepsilon > 0$ ,  $\varepsilon' > 0$  and  $k' \in \mathbb{R}$ , we put

$$\begin{aligned} &Q_b(L_\varepsilon : k' + \varepsilon') \\ &:= \left\{ f(\zeta) \in \mathcal{H}(\overset{\circ}{L}_\varepsilon) \cap \mathcal{C}(L_\varepsilon) : \sup_{\zeta \in L_\varepsilon} |f(\zeta) e^{k'\xi + \varepsilon'|\xi|}| < \infty, \quad \zeta = \xi + i\eta \right\}. \end{aligned}$$

$\mathcal{H}(\overset{\circ}{L}_\varepsilon)$  is the space of holomorphic functions defined on  $\overset{\circ}{L}_\varepsilon$ , (interior of  $L_\varepsilon$ ).  $\mathcal{C}(L_\varepsilon)$  is the space of continuous functions defined on  $L_\varepsilon$ . We put

$$Q(L : k') = \varinjlim_{\varepsilon \rightarrow 0, \varepsilon' \rightarrow 0} Q_b(L_\varepsilon : k' + \varepsilon'),$$

where  $\varinjlim$  means inductive limit. If  $z \in (-k', \infty) + i\mathbb{R}$ , then the function  $e^{\zeta z}$  of  $\zeta$  belongs to  $Q(L : k')$ . We denote by  $Q'(L : k')$  the dual space of  $Q(L : k')$ . The elements of  $Q'(L : k')$  is called analytic functionals with carrier  $L$  and of type  $k'$ .

We define the Fourier-Borel transform  $\tilde{T}(z)$  of  $T \in Q'(L : k')$  as follows:

$$\tilde{T}(z) = \langle T_\zeta, e^{\zeta z} \rangle.$$

$\tilde{T}(z)$  is holomorphic function on the right half plane  $(-k', \infty) + i\mathbb{R}$  and satisfies following estimate:

$\forall \varepsilon > 0, \varepsilon' > 0, \exists C_{\varepsilon, \varepsilon'} \geq 0$  such that

$$|\tilde{T}(z)| \leq C_{\varepsilon, \varepsilon'} e^{ax+b|y|+\varepsilon|z|}, \quad (\operatorname{Re} z \geq -k' + \varepsilon', \quad z = x + iy). \quad (1)$$

$\operatorname{Exp}((-k', \infty) + i\mathbb{R} : L)$  denotes the space of holomorphic functions defined on the right half plane  $(-k', \infty) + i\mathbb{R}$  and satisfy the estimates (1). Following theorem characterizes the Fourier-Borel transform of  $Q'(L : k')$ .

**Theorem 1.1 ([3]).** *Fourier-Borel transform is a linear topological isomorphism from  $Q'(L : k')$  onto  $\operatorname{Exp}((-k', \infty) + i\mathbb{R} : L)$ .*

## 2 Plana's summation formula for holomorphic functions of exponential type.

In this section, we will derive Plana's summation formula for holomorphic functions of exponential type by using the theory of analytic functionals with unbounded carrier.

**Proposition 2.1.** *Let  $T \in Q'(L : k')$ . If  $k' > 0$ ,  $0 \leq b < 2\pi$ ,  $\text{Res} > a$  and  $|\text{Im}s| + b < 2\pi$ , then we have*

$$\begin{aligned} & \sum_{n=0}^{\infty} \tilde{T}(n) e^{-sn} \\ &= \frac{1}{2} \tilde{T}(0) + \int_0^{\infty} \tilde{T}(x) e^{-sx} dx + i \int_0^{\infty} \frac{\tilde{T}(ix) e^{-isx} - \tilde{T}(-ix) e^{isx}}{e^{2\pi x} - 1} dx. \end{aligned}$$

In order to prove proposition 2.1, we prepare the following lemmas.

**Lemma 2.2.** *Assume that  $\text{Res} > a$ . Then the function  $\frac{1}{1 - e^{\zeta-s}}$  of  $\zeta$  belongs to  $Q(L : k')$  and*

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N e^{(\zeta-s)n} = \frac{1}{1 - e^{\zeta-s}} \quad \text{in } Q(L : k').$$

**Lemma 2.3.** *If  $|\text{Im}(\zeta - s)| < 2\pi$ , we have*

$$\frac{1}{e^{\zeta-s} - 1} = \frac{-1}{\zeta - s} - \frac{1}{2} + 2 \int_0^{\infty} \frac{\sin(\zeta - s)x}{e^{2\pi x} - 1} dx.$$

**Proof.** Following equality is well known:

$$\frac{1}{e^{\zeta-s} - 1} = \frac{1}{2i} \sum_{n=1}^{\infty} \left\{ \frac{1}{2\pi n - i(\zeta - s)} - \frac{1}{2\pi n + i(\zeta - s)} \right\}.$$

Therefore we have

$$\begin{aligned} \frac{1}{e^{\zeta-s} - 1} &= \frac{1}{2i} \sum_{n=1}^{\infty} \left\{ \frac{1}{2\pi n - i(\zeta - s)} - \frac{1}{2\pi n + i(\zeta - s)} \right\} \\ &= \frac{1}{2i} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\{2\pi n - i(\zeta - s)\}x} - e^{-\{2\pi n + i(\zeta - s)\}x} dx \\ &= \lim_{N \rightarrow \infty} \int_0^{\infty} \sum_{n=1}^N e^{-2\pi nx} \sin(\zeta - s)x dx. \end{aligned} \tag{2}$$

**Proof.** By proposition 2.4 and Fubini's theorem, we obtain the lemma 2.5. ■

**Lemma 2.6.**

$$\left\langle T_\zeta, \int_0^\infty \frac{e^{-\imath(\zeta-s)x} - e^{\imath(\zeta-s)x}}{e^{2\pi x} - 1} dx \right\rangle = \int_0^\infty \frac{\tilde{T}(-\imath x)e^{\imath s x} - \tilde{T}(\imath x)e^{-\imath s x}}{e^{2\pi x} - 1} dx.$$

**Proof.** By proposition 2.4 and Fubini's theorem, we obtain the lemma 2.6. ■

**Proof of Proposition 2.1.** Since  $\tilde{T}(n) = \langle T_\zeta, e^{\zeta n} \rangle$ ,

$$\begin{aligned} \sum_{n=0}^{\infty} \tilde{T}(n) e^{-sn} &= \sum_{n=0}^{\infty} \langle T_\zeta, e^{\zeta n} \rangle e^{-sn} \\ &= \left\langle T_\zeta, \frac{1}{1 - e^{\zeta-s}} \right\rangle, \end{aligned} \quad (3)$$

By lemma 2.3, we have

$$\frac{1}{1 - e^{\zeta-s}} = \frac{1}{2} + \frac{1}{\zeta - s} + \imath \int_0^\infty \frac{e^{-\imath(\zeta-s)x} - e^{\imath(\zeta-s)x}}{e^{2\pi x} - 1} dx.$$

By lemma 2.6, we have

$$\begin{aligned} (3) &= \left\langle T_\zeta, \frac{1}{2} + \frac{1}{\zeta - s} + \imath \int_0^\infty \frac{e^{-\imath(\zeta-s)x} - e^{\imath(\zeta-s)x}}{e^{2\pi x} - 1} dx \right\rangle \\ &= \left\langle T_\zeta, \frac{1}{2} \right\rangle + \left\langle T_\zeta, \frac{1}{\zeta - s} \right\rangle \\ &\quad + \imath \int_0^\infty \left\{ \frac{\langle T_\zeta, e^{-\imath(\zeta-s)x} \rangle - \langle T_\zeta, e^{\imath(\zeta-s)x} \rangle}{e^{2\pi x} - 1} \right\} dx \\ &= \frac{1}{2} \tilde{T}(0) + \int_0^\infty \tilde{T}(x) e^{-sx} dx \\ &\quad + \imath \int_0^\infty \left\{ \frac{\langle T_\zeta, e^{-\imath(\zeta-s)x} \rangle - \langle T_\zeta, e^{\imath(\zeta-s)x} \rangle}{e^{2\pi x} - 1} \right\} dx. \\ &= \frac{1}{2} \tilde{T}(0) + \int_0^\infty \tilde{T}(x) e^{-sx} dx + \imath \int_0^\infty \frac{\tilde{T}(-\imath x) e^{\imath s x} - \tilde{T}(\imath x) e^{-\imath s x}}{e^{2\pi x} - 1} dx. \quad \blacksquare \end{aligned}$$

We put

$$f_N(x) = \sum_{n=1}^N e^{-2\pi nx} \sin(\zeta - s)x,$$

$$f(x) = \left| \frac{\sin(\zeta - s)x}{e^{2\pi x} - 1} \right|,$$

then we have

$$|f_N(x)| \leq |f(x)| \quad (x \geq 0),$$

$$f(x) \in L^1([0, \infty)).$$

Therefore by Lebesgue's dominated convergence theorem, we have

$$(2) = \int_0^\infty \sum_{n=1}^\infty e^{-2\pi nx} \sin(\zeta - s)x dx$$

$$= \int_0^\infty \frac{\sin(\zeta - s)x}{e^{2\pi x} - 1} dx. \quad \blacksquare$$

**Proposition 2.4** ([4]). *Let  $T \in Q'(L : k')$ . For  $\forall \varepsilon > 0$  and  $\forall \varepsilon' > 0$ , there exists a finite measure  $\mu$  on  $L_\varepsilon$  such that*

$$\langle T, \varphi \rangle = \int_{L_\varepsilon} \varphi(\zeta) e^{k'\xi + \frac{\varepsilon'}{2}|\xi|} d\mu(\zeta) \quad (\zeta = \xi + i\eta),$$

$$\text{for } \varphi(\zeta) \in Q_b(L_\varepsilon : k' + \varepsilon').$$

If  $k' > 0$  and  $s \notin L$ , then the function  $\frac{1}{\zeta - s}$  of  $\zeta$  belongs to  $Q(L : k')$ .

So we can define the Cauchy-Hilbert transform  $\tilde{T}(s)$  of  $T \in Q'(L : k')$  as follows:

$$\tilde{T}(s) = \left\langle T_\zeta, \frac{1}{\zeta - s} \right\rangle.$$

**Lemma 2.5.** *If  $\text{Res} > a$ , we have*

$$\tilde{T}(s) = \int_0^\infty \tilde{T}(x) e^{-xs} dx.$$

**Theorem 2.7.** Suppose that  $f(z) \in \text{Exp}((-k', \infty) + i\mathbb{R} : L)$ . If  $0 \leq b < 2\pi$ ,  $|\text{Im}s| + b < 2\pi$ ,  $\text{Re}s > a$  and  $k' > 0$ , then following equality holds:

$$\begin{aligned} \sum_{n=0}^{\infty} f(n)e^{-sn} \\ = \frac{1}{2}f(0) + \int_0^{\infty} f(x)e^{-sx}dx + i \int_0^{\infty} \frac{f(ix)e^{-isx} - f(-ix)e^{isx}}{e^{2\pi x} - 1} dx. \end{aligned} \quad (4)$$

**Proof.** By theorem 1.1, there exists  $T \in Q'(L : k')$  such that  $f(z) = \tilde{T}(z)$ . By proposition 2.1, we obtain (4). ■

**Corollary 2.8.** Suppose that  $f(z)$  is a holomorphic function on the right half plane  $(-k', \infty) + i\mathbb{R}$ ,  $k' > 0$  and satisfies

$$\forall \varepsilon' > 0, \exists C_{\varepsilon'} \geq 0, \quad |f(z)| \leq C_{\varepsilon'} \frac{e^{b|\text{Im}z|}}{(1+|x|)^m}, \quad (\text{Re}z \geq -k' + \varepsilon', \quad z = x + iy).$$

If  $m > 1$  and  $0 \leq b < 2\pi$ , then we have

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(t)dt + i \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt. \quad (5)$$

**Proof.** From the assumption  $m > 1$ ,  $\sum_{n=0}^{\infty} f(n)$  converges absolutely and  $f(t) \in L^1[0, \infty)$ . So we can apply Lebesgue dominated convergence theorem. Letting  $s \rightarrow 0$  in (4), we obtain (5). ■

**Corollary 2.9.** Let  $f(z)$  be a holomorphic function and  $\text{Im}f(z) \geq 0$  on  $(-k', \infty) + i\mathbb{R}$ ,  $k' > 0$ . If  $\text{Re}s > 0$  and  $|\text{Im}s| < 2\pi$ , we have

$$\sum_{n=0}^{\infty} f(n)e^{-ns} = \frac{1}{2}f(0) + \int_0^{\infty} f(x)e^{-xs}dx + i \int_0^{\infty} \frac{f(ix)e^{-isx} - f(-ix)e^{isx}}{e^{2\pi x} - 1} dx.$$

**Proof.** In order to prove corollary 2.9, we use the following lemma:

**Lemma 2.10 ([5]).** Let  $f(z)$  be a holomorphic and  $\text{Im}f(z) \geq 0$  on the upper half plane  $\mathbb{R} + i\mathbb{R}^+ := \{z \in \mathbb{C} : z = x + iy, \quad y > 0\}$ . Then it satisfies the estimate

$$|f(z)| \leq \sqrt{2}|f(i)| \frac{1+|z|^2}{|y|} \quad z \in \mathbb{R} + i\mathbb{R}^+. \quad (6)$$

Therefore by (6) and theorem 2.7, we obtain the corollary 2.9. ■

**Proposition 2.11.** *Let  $T \in Q'(L : k')$ . If  $k' > 0$ ,  $0 \leq b < 2\pi$ ,  $\text{Res} > a$  and  $|\text{Im}s| + b < 2\pi$ , then we have*

$$\begin{aligned} \sum_{n=0}^{\infty} \tilde{T}(n) e^{-sn} \\ = \frac{1}{2} \tilde{T}(0) + \int_0^{\infty} \tilde{T}(x) e^{-sx} dx + \sum_{n=1}^{\infty} (\tilde{T}(s + 2\pi i n) - \tilde{T}(s - 2\pi i n)). \end{aligned}$$

**Proof.** By proposition 2.1 and lemma 2.5, we obtain proposition 2.11. ■

### 3 Examples and applications.

Plana's summation formula is very useful in the theory of the special functions [2], [7]. In this section we will give some examples.

**Example 3.1.** For

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(t) dt + i \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt,$$

we put  $f(z) = e^{-Az}$  ( $\text{Re}A > 0$ ,  $|\text{Im}A| < 2\pi$ ). Then

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-An} &= \frac{1}{2} + \int_0^{\infty} e^{-At} dt + i \int_0^{\infty} \frac{e^{-iAt} - e^{iAt}}{e^{2\pi t} - 1} dt \\ \frac{1}{1 - e^{-A}} &= \frac{1}{2} + \frac{1}{A} + i \int_0^{\infty} \frac{e^{-iAt} - e^{iAt}}{e^{2\pi t} - 1} dt \\ &= \frac{1}{2} + \frac{1}{A} + 2 \int_0^{\infty} \frac{\sin At}{e^{2\pi t} - 1} dt. \end{aligned}$$

Here we have

$$\begin{aligned} \int_0^{\infty} \frac{e^{-iAt} - e^{iAt}}{e^{2\pi t} - 1} dt &= \int_0^{\infty} (e^{-iAt} - e^{iAt}) \sum_{n=1}^{\infty} e^{-2\pi nt} dt \\ &= \sum_{n=1}^{\infty} \left( \frac{1}{2\pi n + iA} - \frac{1}{2\pi n - iA} \right). \end{aligned}$$



Hence we have

$$\frac{1}{1 - e^{-A}} = \frac{1}{2} + \frac{1}{A} + \sum_{n=1}^{\infty} \left( \frac{1}{A - 2i\pi n} + \frac{1}{A + 2i\pi n} \right).$$

**Example 3.2.** Let  $f(z) \equiv 1$ , i.e.  $f(z) = \langle \delta_z, e^{\zeta z} \rangle$ . Then

$$\sum_{n=0}^{\infty} f(n)e^{-sn} = \frac{1}{2}f(0) + \int_0^{\infty} f(t)e^{-st}dt + i \int_0^{\infty} \frac{f(it)e^{-ist} - f(-it)e^{ist}}{e^{2\pi t} - 1} dt$$

becomes

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-sn} &= \frac{1}{2} + \int_0^{\infty} e^{-st}dt + i \int_0^{\infty} \frac{e^{-ist} - e^{ist}}{e^{2\pi t} - 1} dt \\ \frac{1}{e^s - 1} + \frac{1}{2} - \frac{1}{s} &= 2 \int_0^{\infty} \frac{\sin st}{e^{2\pi t} - 1} dt. \end{aligned}$$

**Example 3.3 (Schlömlich expansion ([6])).** Let  $f(z) = J_0(za)$  ( $J_0(z)$  is Bessel function with degree 0,  $a > 0$ ,  $s > 0$ ). Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} J_0(na)e^{-ns} \\ = \frac{1}{\sqrt{a^2 + s^2}} - \frac{1}{2} + \sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{a^2 + (s + 2i\pi n)^2}} - \frac{1}{\sqrt{a^2 + (s - 2i\pi n)^2}} \right\}. \end{aligned}$$

**Proof.** We have

$$\begin{aligned} \int_0^{\infty} f(t)e^{-st}dt &= \int_0^{\infty} J_0(ta)e^{-st}dt \\ &= \frac{1}{\sqrt{s^2 + a^2}}. \end{aligned}$$

By proposition 2.11, we obtain example 3.3. ■

**Example 3.4.** Let  $l \in \mathbb{N} \setminus \{0\}$  and  $f(z) = J_l(za)$  ( $J_l(z)$  is Bessel function with degree  $l$ ,  $a > 0$ ,  $s > 0$ ). Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} J_l(na) e^{-sn} &= \frac{(-s + \sqrt{a^2 + s^2})^l}{a^l \sqrt{a^2 + s^2}} \\ &+ \sum_{n=1}^{\infty} \left\{ \frac{\left(-s - 2\pi in + \sqrt{a^2 + (s + 2\pi in)^2}\right)^l}{a^l \sqrt{a^2 + (s + 2\pi in)^2}} \right. \\ &\quad \left. - \frac{\left(-s + 2\pi in - \sqrt{a^2 + (s - 2\pi in)^2}\right)^l}{a^l \sqrt{a^2 + (s - 2\pi in)^2}} \right\}. \end{aligned}$$

**Proof.** We have

$$\int_0^{\infty} J_l(ta) e^{-st} dt = \frac{(-s + \sqrt{a^2 + s^2})^l}{a^l \sqrt{a^2 + s^2}}.$$

We obtain example 3.4. ■

**Example 3.5 (Hermite's Formula for Hurwitz Zeta Function ([2])).**

Let

$$f_a(\zeta) = \frac{1}{(\zeta + a)^z}, \quad a \text{ is a positive constant, } \operatorname{Re} z > 1.$$

Then we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(n + a)^z} \\ = \frac{1}{2a^z} + \frac{a^{1-z}}{z-1} + 2 \int_0^{\infty} \frac{\sin(z \arctan(\frac{t}{a}))}{(e^{2\pi t} - 1)(a^2 + t^2)^{\frac{z}{2}}} dt \quad (\operatorname{Re} z > 1). \end{aligned}$$

**Proof.** We have

$$\int_0^{\infty} \frac{(-it + a)^z - (it + a)^z}{(e^{2\pi t} - 1)(a^2 + t^2)^{\frac{z}{2}}} dt = 2 \int_0^{\infty} \frac{\sin(z \arctan(\frac{t}{a}))}{(e^{2\pi t} - 1)(a^2 + t^2)^{\frac{z}{2}}} dt.$$

Therefore we have example 3.5. ■

**Example 3.6 (Appell's Function ([2])).**

$$f(\zeta) = \frac{b^\zeta}{(\zeta + a)^z}, \quad (a, b \in \mathbb{R}, \quad 0 < b < 1, \quad \operatorname{Re} z > 1, \quad a > 0).$$

Then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{b^n}{(n+a)^z} \\ &= \frac{1}{2a^z} + \int_0^\infty \frac{b^t}{(t+a)^z} dt + 2 \int_0^\infty \frac{\sin(z \arctan(\frac{t}{a}) - t \log b)}{(e^{2\pi t} - 1)(a^2 + t^2)^{\frac{z}{2}}} dt. \end{aligned}$$

**Proof.** We have

$$i \int_0^\infty \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt = 2 \int_0^\infty \frac{\sin(z \arctan(\frac{t}{a}) - t \log b)}{(e^{2\pi t} - 1)(a^2 + t^2)^{\frac{z}{2}}} dt.$$

Therefore we have example 3.6. ■

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